

MAS231 Geometry II

Test

21 February 2008

The duration of this test is 40 minutes. Write your name and student number in the spaces below.

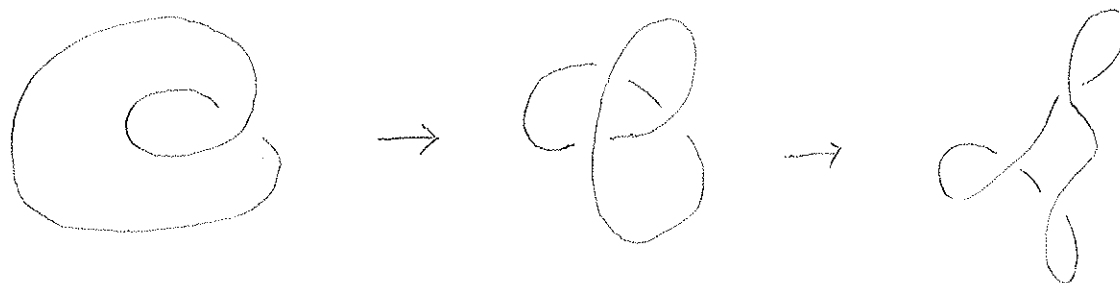
Electronic calculators may be used. The calculator must not be preprogrammed prior to the examination.

Answer all questions. Write all answers in the spaces provided. If you run out of space for an answer, continue on the back of the page.

Name: _____

Student Number: _____

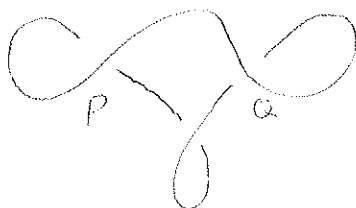
1.



- (a) 8marks Which two Reidemeister moves are used in the two transitions between the three knot diagrams above?

Reidemeister Move II followed by
Reidemeister Move III

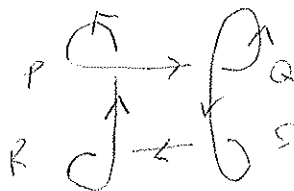
- (b) 14marks By using the definition of tricolouring, prove that no tricolouring of the knot diagram below exists.



Suppose crossing P has only one colour.
Then crossing Q has at most two colours
and so has only one colour. Therefore
all arcs are coloured the same colour,
which is not allowed.

It is impossible for crossing P to have all
three colours, because one arc appears
twice at crossing P.
Therefore, the knot diagram is not tricolourable.

- (c) 8marks Find the writhe of the knot diagram below.



$$\text{sgn}(P) = 1 \quad \text{sgn}(Q) = 1, \quad \text{sgn}(S) = -1, \quad \text{sgn}(R) = -1$$

$$\text{writhe} = 1 + 1 - 1 - 1 = 0$$

2.

- (a) 6marks State the inductive definition of the Kauffman bracket B .

$$1. B(\bigcirc) = 1$$

$$2. B\left(\begin{array}{c} \text{crossing} \\ \text{with box} \end{array}\right) = x B\left(\begin{array}{c} \text{crossing} \\ \text{without box} \end{array}\right) + x^{-1} B\left(\begin{array}{c} \text{crossing} \\ \text{with box} \end{array}\right)$$

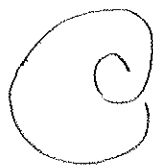
$$B\left(\begin{array}{c} \text{crossing} \\ \text{with box} \end{array}\right) = x^{-1} B\left(\begin{array}{c} \text{crossing} \\ \text{without box} \end{array}\right) + x B\left(\begin{array}{c} \text{crossing} \\ \text{with box} \end{array}\right)$$

$$3. B(\square \bigcirc) = -(x^2 + x^{-2}) B(\square)$$

- (b) 8marks Prove that the Kauffman Bracket is not invariant under Reidemeister Move I.

$$\begin{aligned} B\left(\begin{array}{c} \text{crossing} \\ \text{with box} \end{array}\right) &= x^{-1} B\left(\begin{array}{c} \text{crossing} \\ \text{without box} \end{array}\right) + x B\left(\begin{array}{c} \text{crossing} \\ \text{with box} \end{array}\right) \\ &= x^{-1} B(\square) + x (-(x^2 + x^{-2})) B(\square) \\ &= (x^{-1} - x^3 - x^{-1}) B(\square) = (-x^3) B(\square) \\ &\neq B(\square). \end{aligned}$$

- (c) 9marks Find the Kauffman bracket of the knot diagram below by using the definition you stated in part (a).



$$\begin{aligned}
 B(\text{trefoil}) &= x B(\text{unknot}) + x^{-1} B(\text{unknot}) \\
 &= x(1) + x^{-1}(-x^2 - x^{-2}) B(0) \\
 &= x - (x + x^{-3})(1) \\
 &= -x^{-3}
 \end{aligned}$$

- (d) 7marks Define the Jones polynomial $J(t)$ of a knot diagram in terms of its writhe W and Kauffman Bracket. Determine the Jones polynomial of the knot diagram in part (c) by using your answer to part (c) or by any other valid way.

$$J(x) = (-x)^{-3W} B(x)$$

By applying ~~move~~ Reidemeister Move I
we transform to .

Therefore $J(\text{trefoil}) = J(0) = 1$.

3. 8marks Let $\gamma(t)$ be a unit speed curve. Prove that $\dot{\gamma}(t)$ and $\ddot{\gamma}(t)$ are orthogonal.

$$\begin{aligned}
 \|\dot{\gamma}\|^2 &= 1 \Rightarrow \dot{\gamma} \cdot \dot{\gamma} = 1 \\
 &\Rightarrow \frac{d}{dt}(\dot{\gamma} \cdot \dot{\gamma}) = \frac{d}{dt}(1) = 0 \\
 &\Rightarrow \dot{\gamma} \cdot \ddot{\gamma} + \ddot{\gamma} \cdot \dot{\gamma} = 0 \\
 &\Rightarrow 2\dot{\gamma} \cdot \ddot{\gamma} = 0 \\
 &\Rightarrow \dot{\gamma} \cdot \ddot{\gamma} = 0
 \end{aligned}$$

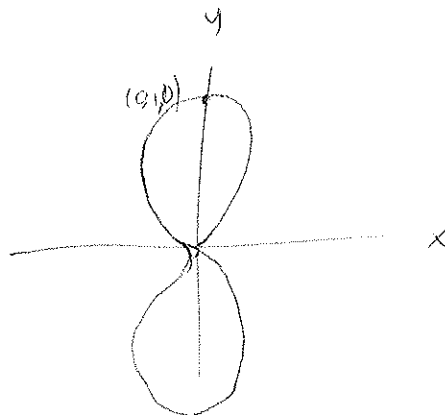
4. 7marks Define what it means for a curve $\tilde{\gamma}(\tilde{t})$ to be a reparametrization of another curve $\gamma(t)$.

It means that for some open interval $(\tilde{\alpha}, \tilde{\beta})$ there exists a smooth bijection $\phi: (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ with smooth inverse such that

$$\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$$

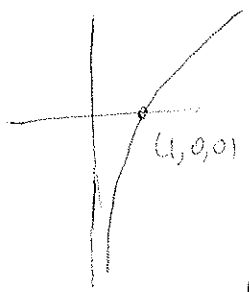
5.

- (a) 5marks Sketch the curve $\gamma(t) = (\sin^2(t)\cos(t), \sin^2(t)\sin(t), 0)$.



- (b) 12marks Sketch the curve $\gamma(t) = (t, \log t, 0)$ for $t \in (0, \infty)$. Find the curvature K and the torsion T for this curve.

How does the curvature behave as $t \rightarrow \infty$ and as $t \rightarrow 0$?



$$\dot{\gamma}(t) = \left(1, \frac{1}{t}, 0\right) \quad \ddot{\gamma}(t) = \left(0, -\frac{1}{t^2}, 0\right)$$

$$\dot{\gamma}(t) \times \ddot{\gamma}(t) = \hat{k} \left(-\frac{1}{t^2}\right)$$

$$K = \frac{\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^3} = \frac{1/t^2}{\left(1 + 1/t^2\right)^{3/2}}$$

$T=0$ because γ is planar.

$$\lim_{t \rightarrow \infty} K = \frac{0}{(1+0)^{3/2}} = 0. \quad \lim_{t \rightarrow 0} K = \lim_{t \rightarrow 0} \frac{t}{(t^2+1)^{3/2}} = \frac{0}{(0+1)^{3/2}} = 0.$$

- (c) 8marks Find the orthonormal vectors $\vec{\tau}, \vec{\nu}, \vec{\beta}$ for the curve $\gamma(t)$ in (b).

$$\vec{\tau} = \frac{\dot{\gamma}}{\|\dot{\gamma}\|} = \frac{1}{\sqrt{1 + \frac{1}{t^2}}} \left(1, \frac{1}{t}, 0\right)$$

$$\vec{\beta} = \frac{\dot{\gamma} \times \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|} = \frac{\hat{k} \left(-1/t^2\right)}{1/t^2} = -\hat{k} = (0, 0, -1)$$

$$\vec{\nu} = \vec{\beta} \times \vec{\tau} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & -1 \\ \frac{1}{\sqrt{1+1/t^2}} & \frac{1/t}{\sqrt{1+1/t^2}} & 0 \end{pmatrix} = \left(\frac{1/t}{\sqrt{1+1/t^2}}, -\frac{1}{\sqrt{1+1/t^2}}, 0 \right)$$

$$= \frac{1}{\sqrt{1+1/t^2}} \left(\frac{1}{t}, -1, 0 \right)$$